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REPRESENTATIONS OF RECIPROCALS OF LUCAS SEQUENCES

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Abstract. In 1953 Stancliff noted an interesting property of the Fibonacci number $F_{11} = 89$. One has that

$$\frac{1}{89} = \frac{F_0}{10} + \frac{F_1}{10^2} + \frac{F_2}{10^3} + \frac{F_3}{10^4} + \frac{F_4}{10^5} + \frac{F_5}{10^6} + \cdots.$$

De Weger determined a complete list of similar identities in case of the Fibonacci sequence, the solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In this article we study similar problems in case of general Lucas sequences $U_n(P,Q)$. We deal with equations of the form

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k},$$

for certain pairs $(P_1,Q_1) \neq (P_2,Q_2)$. We also consider equations of the form

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k},$$

where R_n is a ternary linear recurrence sequence. The proofs are based on results related to Thue equations and elliptic curves.

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1. INTRODUCTION

Let P and Q be non-zero relatively prime integers. The Lucas sequence $\{U_n(P,Q)\}$ is defined by

$$U_0 = 0, U_1 = 1 \text{ and } U_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

The associated Lucas sequence $\{V_n(P,Q)\}$ is defined by

$$V_0 = 2, V_1 = P \text{ and } V_n = PU_{n-1} - QU_{n-2}, \text{ if } n \geq 2.$$

Terms of Lucas sequences and associated Lucas sequences satisfy the identity

$$V_n^2 - DU_n^2 = 4Q^n, \tag{1.1}$$

where $D = P^2 - 4Q$. In 1953, Stancliff [12] noted an interesting property of the Fibonacci sequence $U_n(1,-1) = F_n$. One has that

$$\frac{1}{F_{11}} = \frac{1}{89} = 0.0112358\ldots = \sum_{k=0}^{\infty} \frac{F_k}{10^{k+1}}.$$

In 1980, Winans [17] studied the related sums

$$\sum_{k=0}^{\infty} \frac{F_{\alpha k}}{10^{k+1}}$$

for certain values of α. In 1981 Hudson and Winans [7] characterized all decimal fractions that can be approximated by sums of the type

$$\frac{1}{F_\alpha} \sum_{k=1}^{n} \frac{F_{\alpha k}}{10^{l(k+1)}}, \quad \alpha, l \geq 1.$$

Long [10] obtained a general identity for binary recurrence sequences from which one obtains e.g.

$$\frac{1}{109} = \sum_{k=0}^{\infty} \frac{F_k}{(-10)^{k+1}}, \qquad \frac{1}{10099} = \sum_{k=0}^{\infty} \frac{F_k}{(-100)^{k+1}}.$$

In case of the equation

$$\frac{1}{U_n(P,Q)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P,Q)}{x^k}, \tag{1.2}$$

De Weger [4] determined all $x \geq 2$ in case of $(P,Q) = (1,-1)$. The solutions are as follows

$$\frac{1}{F_1} = \frac{1}{F_2} = \frac{1}{1} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{2^k}, \qquad \frac{1}{F_5} = \frac{1}{5} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{3^k},$$

$$\frac{1}{F_{10}} = \frac{1}{55} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{8^k}, \qquad \frac{1}{F_{11}} = \frac{1}{89} = \sum_{k=1}^{\infty} \frac{F_{k-1}}{10^k}.$$

In 2014 Tengely [15] extended the above result and obtained e.g.

$$\frac{1}{U_{10}} = \frac{1}{416020} = \sum_{k=0}^{\infty} \frac{U_k}{647^{k+1}},$$

where $U_0 = 0, U_1 = 1$ and $U_n = 4U_{n-1} + U_{n-2}, n \geq 2$. Recently Ohtsuka and Nakamura [11] proved that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \geq 2 \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \geq 1 \text{ is odd,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. This result has been investigated by several other mathematicians see e.g. [6, 9].

2. Auxiliary results

In the proofs we will use the following two results of Köhler [8].

Lemma 1. *Let A, B, a_0, a_1 be arbitrary complex numbers. Define the sequence $\{a_n\}$ by the recursion $a_{n+1} = Aa_n + Ba_{n-1}$. Then the formula*

$$\sum_{k=0}^{\infty} \frac{a_k}{x^{k+1}} = \frac{a_0 x - Aa_0 + a_1}{x^2 - Ax - B}$$

holds for all complex x such that $|x|$ is larger than the absolute values of the zeros of $x^2 - Ax - B$.

Lemma 2. *Let arbitrary complex numbers $A_0, A_1, \ldots, A_m, a_0, a_1, \ldots, a_m$ be given. Define the sequence $(a_n)_n$ by the recursion*

$$a_{n+1} = A_0 a_n + A_1 a_{n-1} + \cdots + A_m a_{n-m}$$

Then for all complex z such that $|z|$ is larger than the absolute values of all zeros of $q(z) = z^{m+1} - A_0 z^m - A_1 z^{m-1} - \cdots - A_m$, the formula

$$\sum_{k=1}^{\infty} \frac{a_{k-1}}{z^k} = \frac{p(z)}{q(z)}$$

holds with $p(z) = a_0 z^m + b_1 z^{m-1} + \cdots + b_m$, where $b_k = a_k - \sum_{i=0}^{k-1} A_i a_{k-1-i}$ for $1 \leq k \leq m$.

3. MAIN RESULTS

In this paper we extend the results of [15], we consider the equation

$$\frac{1}{U_n(P_2,Q_2)} = \sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k}, \tag{3.1}$$

for certain pairs $(P_1,Q_1) \neq (P_2,Q_2)$. We consider non-degenerate sequences with $1 \leq P \leq 3$ and $Q = \pm 1$. Define the set S as follows

$$S = \{u_1(n) = U_n(1,-1), u_2(n) = U_n(1,1), u_3(n) = U_n(2,-1), u_4(n) = U_n(3,-1), u_5(n) = U_n(3,1)\}.$$

Theorem 1. *The equation*

$$\frac{1}{u_j(n)} = \sum_{k=1}^{\infty} \frac{u_i(k-1)}{x^k}, \tag{3.2}$$

has the following solutions with $1 \leq i,j \leq 5, i \neq j$

$$\begin{aligned}(i,j,n,x) \in \{&(1,2,\{1,2\},2),(1,3,1,2),(1,3,3,3),(1,3,5,6),(1,4,1,2),\\ &(1,4,5,11),(1,4,7,35),(1,5,1,2),(1,5,5,8),(2,1,4,2),(2,1,7,4),\\ &(2,1,8,5),(2,5,2,2),(2,5,4,5),(3,1,3,3),(3,1,9,7),(4,1,4,4),\\ &(4,1,14,21),(4,5,2,4),(4,5,7,21),(5,1,\{1,2\},3),(5,1,5,4),\\ &(5,1,10,9),(5,1,11,11),(5,2,\{1,2\},3),(5,3,1,3),(5,3,3,4),\\ &(5,3,5,7),(5,4,1,3),(5,4,5,12),(5,4,7,36)\}.\end{aligned}$$

We also deal with equations of the form

$$\sum_{k=1}^{\infty} \frac{u_j(k-1)}{x^k} = \sum_{k=1}^{\infty} \frac{R_{k-1}}{y^k}, \tag{3.3}$$

where R_n is a ternary linear recurrence sequence. We provide results in case of the Tribonacci sequence defined by $T_0 = T_1 = 0, T_2 = 1$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n, n \geq 0$ and Berstel's sequence, that is given by $B_0 = B_1 = 0, B_2 = 1$ and $B_{n+3} = 2B_{n+2} - 4B_{n+1} + 4B_n, n \geq 0$.

Theorem 2. *The complete list of solutions of equation (3.3) with $u_n \in S, R_n \in \{B_n, T_n\}$ and positive integers x,y satisfying conditions of Lemma 1 and 2 is as follows*

u_n	R_n	(x,y)	u_n	R_n	(x,y)
u_1	B_n	$\{(25,9)\}$	u_1	T_n	$\{(2,2)\}$
u_2	B_n	$\{(10,5)\}$	u_2	T_n	$\{(7,4),(309,46)\}$
u_3	B_n	$\{\}$	u_3	T_n	$\{(t(t^2-2)+1,t^2-1) : t \geq 2, t \in \mathbb{N}\}$
u_4	B_n	$\{(6,3),(18,7)\}$	u_4	T_n	$\{\}$
u_5	B_n	$\{(26,9)\}$	u_5	T_n	$\{\}$

4. Proofs of the theorems

Proof of Theorem 1. Consider equation (3.1), by Lemma 1 we obtain that

$$\sum_{k=1}^{\infty} \frac{U_{k-1}(P_1,Q_1)}{x^k} = \frac{1}{x^2 - P_1x + Q_1}.$$

Hence we have that $U_n(P_2,Q_2) = x^2 - P_1x + Q_1$. Combining the latter equation with (1.1) we get $V_n(P_2,Q_2)^2 = (P_2^2-4Q_2)(x^2 - P_1x + Q_1)^2 + 4Q_2^n$. The so-called two-cover descent by Bruin and Stoll [3] can be used to prove that a given hyperelliptic curve has no rational points. It is implemented in Magma [2], the procedure is called `TwoCoverDescent`. If it fails and we do not find any rational points on the curve, then we apply the argument by Alekseyev and Tengely [1], that reduces the problem to Thue equations. If we have a rational point on the curve, then using a method by Tzanakis [16] the integral points can be determined. This algorithm is implemented in Magma as `IntegralQuarticPoints`. In this way we collect the possible values of x.

(P_1,Q_1,P_2,Q_2)	x	(P_1,Q_1,P_2,Q_2)	x	(P_1,Q_1,P_2,Q_2)	x
$(1,-1,1,1)$	2	$(1,1,1,-1)$	$2,4,5$	$(2,-1,1,-1)$	$3,7$
$(1,-1,2,-1)$	$2,3,6$	$(1,1,2,-1)$	$-$	$(2,-1,1,1)$	$-$
$(1,-1,3,-1)$	$2,11,35$	$(1,1,3,-1)$	2	$(2,-1,3,-1)$	$-$
$(1,-1,3,1)$	$2,8$	$(1,1,3,1)$	$2,5$	$(2,-1,3,1)$	$-$

(P_1,Q_1,P_2,Q_2)	x	(P_1,Q_1,P_2,Q_2)	x
$(3,-1,1,-1)$	$4,21$	$(3,1,1,-1)$	$3,4,9,11$
$(3,-1,1,1)$	$-$	$(3,1,1,1)$	3
$(3,-1,2,-1)$	$-$	$(3,1,2,-1)$	$3,4,7$
$(3,-1,3,1)$	$4,21$	$(3,1,3,-1)$	$3,12,36$

It remains to compute the set of possible values of n. We provide details of the computation in case of $(P_1,Q_1,P_2,Q_2) = (3,-1,1,-1)$, following these steps all other equations can be handled. In case of $(P_1,Q_1,P_2,Q_2) = (3,-1,1,-1)$ we have that $x \in \{4,21\}$. If $x = 4$, then we define a matrix T as follows

$$T = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 0 \end{pmatrix}.$$

We have that

$$\frac{1}{4}\left(T^0+T^1+T^2+\cdots+T^{N-1}\right)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}*\\ \sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}\end{pmatrix}.$$

It follows that

$$\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}=$$
$$-\frac{2^{-3N-1}}{39}\left(\left(\sqrt{13}+3\right)^N\left(5\sqrt{13}+13\right)+\left(13-5\sqrt{13}\right)\left(-\sqrt{13}+3\right)^N-13\cdot 2^{3N+1}\right),$$

hence we have that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{4^k}=\frac{1}{3}=\frac{1}{U_4(1,-1)}.$$

In this case we obtain that $n=4$. If $x=21$, then

$$T=\begin{pmatrix}3/21 & 1/21\\ 1/21 & 0\end{pmatrix}.$$

In a similar way than in case of $x=4$ we get that

$$\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{21^k}=$$
$$\frac{\left(7^N3^N2^{N+1}-\left(\sqrt{13}+3\right)^N\left(3\sqrt{13}+1\right)+\left(3\sqrt{13}-1\right)\left(-\sqrt{13}+3\right)^N\right)2^{-N-1}}{377\cdot 7^N3^N},$$

therefore

$$\lim_{N\to\infty}\sum_{k=1}^{N}\frac{U_{k-1}(3,-1)}{21^k}=\frac{1}{377}=\frac{1}{U_{14}(1,-1)}.$$

The only solution in this case is given by $n=14$. □

Proof of Theorem 2. We provide a general argument that works for other sequences as well. Let $a_0=0,a_1=1$ and $a_{n+1}=Aa_n+Ba_{n-1}$. Let $b_0=b_1=0,b_2=1$ and $b_{n+1}=Cb_n+Db_{n-1}+Eb_{n-2}$. Equation (3.3) yields that

$$Y^2=X^3-4CX^2-16DX+16A^2+64B-64E,$$

where $Y=8x-4A$ and $X=4y$. If the cubic polynomial in X is square-free, then we have an elliptic equation and integral points can be determined using the so-called elliptic logarithm method developed by Stroeker and Tzanakis [14] and independently by Gebel, Pethő and Zimmer [5]. There exists a number of software implementations for determining integral points on elliptic curves based on this technique, here we used SageMath [13]. Let us consider the case with $u_2(n),T_n$. We obtain the elliptic curve $Y^2 = X^3 - 4X^2 - 16X - 112$. Using the SageMath function `integral_points()` we get

$$[(8:4:1),(16:52:1),(29:143:1),(184:2468:1)].$$

From these points we have that $(x,y) \in \{(7,4),(309,46)\}$. As a second example consider the case with u_4, B_n. The elliptic curve is given by $Y^2 = X^3 - 8X^2 + 64X - 48$. The list of integral points is

$$[(1:3:1),(4:12:1),(12:36:1),(28:132:1)].$$

Thus we get that $(x,y) \in \{(6,3),(18,7)\}$. Finally let us deal with the special case with u_3, T_n. The cubic polynomial is not square-free, it is $(X+4)(X-4)^2$. Therefore we have that $X + 4 = 4y + 4 = u^2$. Hence $y = t^2 - 1$ for some integer $t \geq 2$. It follows that $x = t(t^2-2)+1$. So we obtain infinitely many identities of the form

$$\sum_{k=1}^{\infty} \frac{u_4(k-1)}{(t(t^2-2)+1)^k} = \sum_{k=1}^{\infty} \frac{T_{k-1}}{(t^2-1)^k}.$$

□

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